CHAPTER 8

BINOMIAL THEOREM

Binomial theorem for any positive integer n

$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + {}^nC_3a^{n-3}b^3 + \ldots\ldots + {}^nC_nb^n$

Recall

1) ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$
2) ${}^nC_r = {}^nC_{n-r}$

   ${}^7C_4 = {}^7C_3 = \dfrac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$

   ${}^8C_6 = {}^8C_2 = \dfrac{8 \times 7}{1 \times 2} = 28$
3) ${}^nC_n = {}^nC_0 = 1$
4) ${}^nC_1 = n$

OBSERVATIONS/ FORMULAS

1) The coefficients ${}^nC_r$ occurring in the binomial theorem are known as binomial coefficients.
2) There are (n+1) terms in the expansion of $(a+b)^n$, ie one more than the index.
3) The coefficient of the terms equidistant from the beginning and end are equal.
4) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots\ldots + {}^nC_nx^n$. (By putting a = 1 and b = x in the expansion of $(a + b)^n$).
5) $(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - {}^nC_3x^3 + \ldots\ldots + (-1)^n\,{}^nC_nx^n$ (By putting a = 1 and b = -x in the expansion of $(a + b)^n$).
6) $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n$ (By putting x = 1 in (4))
7) $0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots\ldots + (-1)^n\,{}^nC_n$. (By putting x = 1 in (5))

8**) $(r + 1)^{th}$ term in the binomial expansion for $(a+b)^n$ is called the general term which is given by

$$\mathbf{T_{r+1} = {}^nC_r\, a^{n-r}\, b^r.}$$

i.e to find 4^{th} term = T_4, substitute r = 3.

9*) **Middle term** in the expansion of $(a+b)^n$

i) If **n is even**, middle term = $\left[\frac{n}{2}+1\right]^{th}$ term.

ii) If **n is odd**, then 2 middle terms are, $\left[\frac{n+1}{2}\right]^{th}$ term and $\left[\frac{n+1}{2}+1\right]^{th}$ term.

10*) To find the **term independent of x or the constant term,** find the coefficient of x^0.(ie put power of x = 0 and find r)

Problems

eg 4** (4 marks)

Ex 8.1

Q 2,4,7,9 (1 mark)

10*, 11*,12* (4 marks)

13**,14** (4 marks)

13**) Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer

Or

$3^{2n+2} - 8n - 9$ is divisible by 64

Solution: $9^{n+1} - 8n - 9 = (1+8)^{n+1} - 8n - 9$

$= {}^{n+1}C_0 + {}^{n+1}C_1 8 + {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + \ldots\ldots. + {}^{n+1}C_{n+1} 8^{n+1} - 8n - 9$

$= 1 + 8n + 8 + 8^2 [\, {}^{n+1}C_2 + {}^{n+1}C_3.8 + \ldots\ldots + 8^{n-1}] - 8n - 9$

(since ${}^{n+1}C_0 = {}^{n+1}C_{n+1} = 1$, ${}^{n+1}C_1 = {}^{n+1}$,

$8^{n+1}/8^2 = 8^{n+1-2} = 8^{n-1}$)

$= 8^2 [\, {}^{n+1}C_2 + {}^{n+1}C_3.8 + \ldots\ldots + 8^{n-1}]$ which is divisible by 64

Problems

eg 5*,6**,7* (4 marks)

eg 8**,9** (6 marks)

Ex 8.2

Q 2,3* (1 mark)

Q 7**,8**,9**,11**,12** (4 marks), 10** (6 marks)

eg 10**,11 (HOT),12 (HOT), 13(HOT), eg 15*,17** (4 marks)

Misc ex

Q 1** (6 mark),2,3(HOT), 8* (4 marks)

Ex 8.2

Q 10**(6 marks)

The coefficients of the (r-1)th , rth and (r+1)th terms in the expansion of $(x+1)^n$ are in the ratio 1: 3 : 5. Find n and r.

Solution

$T_{r+1} = {}^nC_r x^{n-r}$

$T_r = T_{(r-1)+1} = {}^nC_{r-1} x^{n-r+1}$

$T_{r-1} = T_{(r-2)+1} = {}^nC_{r-2}\, x^{n-r+2}$

Given ${}^nC_{r-2} : {}^nC_{r-1} : {}^nC_r :: 1 : 3 : 5$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3}$$

$$\frac{n!}{(n-r+2)!\,(r-2)!} \div \frac{n!}{(n-r+1)!\,(r-1)!} = \frac{1}{3}$$

$$\frac{(n-r+1)!}{(n-r+2)!} \times \frac{(r-1)!}{(r-2)!} = \frac{1}{3}$$

$$\frac{(n-r+1)!}{(n-r+1)!(n-r+2)} \times \frac{(r-2)!(r-1)}{(r-2)!} = \frac{1}{3}$$

$$\frac{r-1}{n-r+2} = \frac{1}{3}$$

$3r-3 = n-r+2$

$n-4r = -5$ _________(1)

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}$$

simplify as above and get the equation $3n - 8r = -3$ ________(2)
solving (1) and (2) we get
$n = 7$ and $r = 3$.

EXTRA/HOT QUESTIONS

1) Using Binomial theorem show that $2^{3n} - 7n - 1$ or $8^n - 7n - 1$ is divisible by 49 where n is a natural number. (4 marks**)
2) Find the coefficient of x^3 in the equation of $(1+2x)^6 (1-x)^7$ (HOT)
3) Find n if the coefficient of 5^{th} , 6^{th} & 7^{th} terms in the expansion of $(1+x)^n$ are in A.P.
4) If the coefficient of x^{r-1}, x^r, x^{r+1} in the expansion of $(1+x)^n$ are in A.P. prove that $n^2 - (4r+1)n + 4r^2 - 2 = 0$. (HOT)
5) If 6^{th},7^{th},8^{th} & 9^{th} terms in the expansion of $(x+y)^n$ are respectively a,b,c &d then show that $\frac{b^2 - ac}{c^2 - bd} = \frac{4a}{3c}$ (HOT)
6) Find the term independent of x in the expansion of $\left[3x^2 - \frac{1}{2x^3}\right]^{10}$ (4 marks*)
7) Using Binomial theorem show that $3^{3n} - 26n - 1$ is divisible by 676. (4 marks**)
8) The 3^{rd},4^{th} & 5^{th} terms in the expansion of $(x+a)^n$ are 84, 280 & 560 respectively. Find the values of x, a and n. (6 marks**)
9) The coefficient of 3 consecutive terms in the expansion of $(1+x)^n$ are in the ratio 3 : 8 :14. Find n. (6 mark**)
10) Find the constant term in the expansion of $(x-1/x)^{14}$
11) Find the middle term(s) in the expansion of

    i) $\left[\frac{x}{a} - \frac{a}{x}\right]^{10}$ ii) $\left[2x - \frac{x^2}{4}\right]^9$

12) If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots..+C_nx^n$
    Prove that $C_1 + 2C_2 + 3C_3 + \ldots\ldots+ nC_n = n.2^{n-1}$

Answers

2) -43
3) n = 7 or 14
6) 76545/8
8) x =1, a=2, n = 7
9) 10
10) -3432
11) i) -252
    ii) $\frac{-63}{32} x^{14}$