Chapter 4

PRINCIPLE OF MATHEMATICAL INDUCTION

INTRODUCTION

To prove certain results or statements in Algebra, that are formulated in terms of n, where n is a natural number, we use a specific technique called principle of mathematical induction (P.M.I)

Steps of P.M.I

Step I - Let p(n): result or statement formulated in terms of n (given question)

Step II – Prove that P(1) is true

Step III – Assume that P(k) is true

Step IV – Using step III prove that P(k+1) is true

Step V - Thus P(1) is true and P(k+1) is true whenever P(k) is true.

Hence by P.M.I, P(n) is true for all natural numbers n

Type I

Eg: Ex 4.1

1) Prove that

   $1+3+3^2+\ldots\ldots\ldots+3^{n-1} = \frac{3^n-1}{2}$

Solution:-

Step I : Let P(n): $1+3+3^2+\ldots\ldots\ldots+3^{n-1} = \frac{3^n-1}{2}$

Step II: P(1):

LHS = 1

RHS = $= \frac{3-1}{2} = \frac{2}{2} = 1$

LHS=RHS

Therefore p(1) is true.

Step III: Assume that P(k) is true

i.e $1+3+3^2+\ldots\ldots\ldots+3^{k-1} = \frac{3^k-1}{2}$ ______(1)

Step IV: we have to prove that P(k+1) is true.

ie to prove that $1+3+3^2+\dots\dots+3^{k-1}+3 = \frac{3^{k+1}-1}{2}$

Proof

LHS $= (1+3+3^2+\dots\dots+3^{k-1}) + 3$

$= \frac{3^k-1}{2} + 3^k$ from eq(1)

$= \frac{3^k - 1 + 2.3^k}{2}$

$= \frac{3.3^k - 1}{2} = \frac{3^{k+1}-1}{2} = \text{RHS}$

Therefore P(k+1) is true

Step V: Thus P(1) is true and P(k+1) is true whenever P(k) is true. Hence by P.M.I, P(n) is true for all natural number n.

Text book

Ex 4.1

Q. 1,2, 3**(HOT), 4, 5*,6*,7,8,9,10*,11**,12,13**,14**,15,16**,17**, eg 1, eg 3

Type 2

Divisible / Multiple Questions like Q. 20**,21,22**,23 of Ex 4.1

eg 4, eg 6**(HOT)

Q 22. Prove that $3^{2n+2}-8n-9$ is divisible by 8 for all natural number n.

Solution

Step I: Let p(n): $3^{2n+2}-8n-9$ is divisible by 8

Step II: P(1): $3^4 - 8 - 9 = 81-17 = 64$ which is divisible by 8

Therefore p(1) is true

Step III: Assume that p(k) is true

i.e $3^{2k+2} - 8k - 9 = 8m$; m is a natural number.

i.e $3^{2k}.9 = 8m+8k+9$

ie $3^{2k} = \frac{8m+8k+9}{9}$ ______(1)

Step IV: To prove that p(k+1) is true.

ie to prove that 3^{2k+4} -8(k+1) –9 is divisible by 8.

Proof: 3^{2k+4}-8k-17 = $3^{2k}.3^4$-8k-17 =$(\frac{8m+8k+9}{9})$ x 3^4 – 8k-17(from eqn (1))

= (8m+8k+9)9-8k-17 = 72m+72k+81-8k-17 = 72m-64k+64 = 8[9m-8k+8] is divisible by 8.

Step V: Thus P(1) is true and P(k+1) is true whenever P(k) is true. hence by P.M.I, P(n) is true for all natural numbers n.

Type III: Problems based on Inequations

Ex 4.1 Q. 18,14, eg 7

(Q 18) Prove that 1+2+3+……+n< $\frac{(2n+1)^2}{8}$

Step I : Let P(n): 1+2+3+……+n< $\frac{(2n+1)^2}{8}$

Step II: P(1): 1< $\frac{9}{8}$ which is true, therefore p(1) is true.

Step III: Assume that P(k) is true.

ie 1+2+3+……+k< $\frac{(2k+1)^2}{8}$ ________(1)

Step IV: We have to prove that P(k+1) is true. ie to

prove that 1+2+3+…..+k+(k+1)< $\frac{(2k+3)^2}{8}$

Proof: Adding (k+1) on both sides of inequation (1)

1+2+3+…..+k+(k+1) < $\frac{(2k+1)^2}{8} + (k+1)$

$= \frac{(4k^2+4k+1)+8k+8}{8}$

$= \frac{4k^2+12k+9}{8}$

$= \frac{(2k+3)^2}{8}$

Therefore $1+2+3+\ldots..+k+(k+1) < \frac{(2k+3)^2}{8}$

P(k+1) is true.

Step V: Thus P(1) is true and P(k+1) is true whenever P(k) is true. Hence by P.M.I, P(n) is true for all natural number n.

HOT/EXTRA QUESTIONS

Prove by mathematical induction that for all natural numbers n.

1) $a^{2n-1} - 1$ is divisible by a-1 (type II)
2) $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is an integer(HOT)
3) $\sin x + \sin 3x + \ldots\ldots + \sin(2n-1)x = \frac{\sin^2 nx}{\sin x}$ (HOT Type 1)
4) $3^{2n-1}+3^n+4$ is divisible by 2 (type II)
5) Let P(n): n^2+n-19 is prime, state whether P(4) is true or false
6) $2^{2n+3} \leq (n+3)!$ (type III)
7) What is the minimum value of natural number n for which $2^n < n!$ holds true?
8) $7^{2n}+2^{3n-3}.3^{n-1}$ is divisible by 25 (type II)

Answers

5) false
7) 4