Chapter-11

Three Dimensional Geometry

Direction cosines of a line are the cosines of the angles made by the line with the positive direct ions of the coordinate axes.

- *If* l, m, n are the direct ion cosines of a line, then $l^2+m^2+n^2=1$
- Direct ion cosines of a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_2}{PQ}$
- Where $PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$
- Direction ratios of a line are the numbers which are proportional to the direct ion cosines of a line.
- If l, m, n are the direct ion cosines and a, b, c are the direct ion ratios of a line

Then, $l=\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m=\frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n=\frac{c}{\sqrt{a^2+b^2+c^2}}$

- Skew lines are lines in space which are neither parallel nor intersecting. They lie in different planes.
- Angle between skew lines is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.
- If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines; and θ is the acute angle between the two lines; then,

$$\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

- Vector equation of a line that passes through the given point whose position vector is $\bar{a}$ and parallel to a given vector $\bar{b}$ is $\bar{r}=\bar{a}+\lambda\bar{b}$

Key Notes

- Equation of a line through a point (x_1, y_1, z_1) and having direct ion cosines $l,\ m,\ n$ is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

- The vector equation of a line which passes through two points whose posit ion vectors are $\bar{a}$ and $\bar{b}$ is $\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$

- Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

- If θ is the acute angle between $\bar{r} = \bar{a}_1 + \lambda\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda\bar{b}_2$ then, $\cos\theta = \left|\dfrac{\bar{b}.\bar{b}_2}{|\bar{b}_1||\bar{b}_2|}\right|$

- If $\dfrac{x-x_1}{l_1} = \dfrac{y-y_1}{m_1} = \dfrac{z-z_1}{n_1}$ and $\dfrac{x-x_2}{l_2} = \dfrac{y-y_2}{m_2} = \dfrac{z-z_2}{n_2}$ are the equations of two lines, then the acute angle between the two lines is given by $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

- Shortest distance between two skew lines is the line segment perpendicular to both the lines.

- Shortest distance between $\bar{r} = \bar{a}_1 + \wedge\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \wedge\bar{b}_2$ $\left|\dfrac{(\bar{b}_1 \times \bar{b}_2).(\bar{a}_2 - \bar{a}_1)}{|\bar{b}_1 \times \bar{b}_2|}\right|$

- Shortest distance between the lines: $\dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b_1} = \dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_1} = \dfrac{y-y_2}{b_1} = \dfrac{z-z_2}{c_1}$ is

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

- Distance between parallel lines $\bar{r} = \bar{a}_1 + \wedge\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \wedge\bar{b}_2$ $\left|\dfrac{(\bar{b}) \times (\bar{a}_2 - \bar{a}_1)}{|\bar{b}_1|}\right|$

- In the vector form, equation of a plane which is at a distance d from the origin, and n ˆ is the unit vector normal to the plane through the origin is $\bar{r}.\hat{n} = d$
- Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane as l, m, n is $lx + my + nz = d$.
- The equation of a plane through a point whose posit ion vector is a and perpendicular to the vector $\bar{N}$ $is(\bar{r} - \bar{a})$. N=0
- Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point $(x_{1,} y_1, z_1)$ is
- $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$
- Equation of a plane passing through three non collinear points $(x_{1,} y_1, z_1)$
- $(x_{2,} y_2, z_2)$and $(x_{3,} y_3, z_3)$ is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$
- Vector equation of a plane that contains three non collinear points having position vectors
- $\bar{a}, \bar{b}$ and $\bar{c}$ is $(\bar{r} - \bar{a}).\left[(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})\right] = 0$
- Equation of a plane that cuts the coordinates axes at $(a,\ 0,\ 0),\ (0,\ b,\ 0)\ and\ (0,\ 0,\ c)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
- Vector equation of a plane that p asses thro ugh the in the section of planes $\bar{r}.\bar{n}_1 = d_1$ and $\bar{r}.\bar{n}_2 = d_2$ is $\bar{r}.(\bar{n}_1 + \lambda \bar{n}_2) = d_1 + \lambda d_2$ where λ is any nonzero constant.
- Cartesian equation of a plane that passes that passes through the intersection of two given planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is $(A_1x + B_1y + C_1z + D_1) + \lambda(A_2x + B_2y + C_2z + D_2 = 0$
- Two lines $\bar{r} = \bar{a}_1 + \lambda \bar{b}_1$and $\bar{r} = \bar{a}_2 + \mu \bar{b}_2$ are coplanar if $(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times - \bar{b}_2) = 0$

- In the Cartesian form above lines passing through the points $A\left(x_1, y_1, z_1\right)$ and $B\left(x_2, y_2 z_2\right)$
- $=\frac{y-y_2}{b_2}=\frac{z-z_2}{C_2}$ are coplanar if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=0$
- In the vector form, if θ is the angle between the two planes, $\bar{r}.\bar{n}_1=d_1$ and $\bar{r}.\bar{n}_2=d_2$, then $\theta=\cos^{-1}\frac{\left|\bar{n}_1.\bar{n}_2\right|}{\left|\bar{n}_1\right|\left|\bar{n}_2\right|}$
- The angle ϕ between the line $\bar{r}=\bar{a}+\lambda\bar{b}$ and the plane $\bar{r}.\hat{n}=d$ $\sin\phi=\left|\frac{\bar{b}.\hat{n}}{\left|\bar{b}\right|\left|\hat{n}\right|}\right|$
- The angle θ between the planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ is given by $\cos\theta=\left|\frac{A_1A_2++B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}\right|$
- The distance of a point whose position vector is $\bar{a}$ from the plane $\bar{r}.\hat{n}=d$ is $\left|d-\bar{a}.\hat{n}\right|$
- The distance from a point $\left(x_1, y_1, z_1\right)$ to the plane Ax + By + Cz + D = 0 is $\left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|$