Short Answer Type Questions

1. For a positive integer n, find the value of $(1+i)^n\left(1-\frac{1}{i}\right)^n$.

Sol. $(1-i)^n\left(1-\frac{1}{i}\right)^n = (1-i)^n\left(1-\frac{i}{i^2}\right)^n = (1-i)^n(1+i)^n = (1-i^2)^n = 2^n$

2. Evaluate $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $n \in N$.

Sol. $\sum_{n=1}^{13}(i^n + i^{n+1}) = \sum_{n=1}^{13}(1+i)i^n$

$= (1+i)(i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13})$

$= (1+i)(i - 1 - i + 1 + i - 1 - i + 1 + i - 1 - i + 1 + i)$

$= (1+i)i = i + i^2 = i - 1$

Alternative method:

$\sum_{n=1}^{13}(i^n + i^{n+1}) = \sum_{n=1}^{13}(1+i)i^n$

$= (1+i)(i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13})$

$= (1+i)\frac{i(i^{13}-1)}{i-1} = (1+i)\frac{i(i-1)}{i-1} = (1+i)i = i + i^2 = i - 1$

3. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$, then find (x, y).

Sol. $x + iy = \left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$

$$= \left(\frac{(1+i)^2}{1-i^2}\right)^3 - \left(\frac{(1-i)^2}{1-i^2}\right)^3 = \left(\frac{1+2i+i^2}{1+1}\right)^3 - \left(\frac{1-2i+i^2}{1+1}\right)^3$$

$$= \left(\frac{2i}{2}\right)^3 - \left(\frac{-2i}{2}\right)^3 = i^3 - (-i^3) = 2i^3 = 0 - 2i$$

$\Rightarrow$ $x = 0$ and $y = -2$

4. If $\frac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.

Sol. $x + iy = \frac{(1+i)^2}{2-i} = \frac{1+2i+i^2}{2-i} = \frac{2i}{2-i} = \frac{2i(2+i)}{(2-i)(2+i)} = \frac{4i+2i^2}{4-i^2}$

$$= \frac{4i-2}{4+1} = \frac{-2}{5} + \frac{4i}{5}$$

$\Rightarrow$ $x = \frac{-2}{5}, y = \frac{4}{5} \Rightarrow x + y = \frac{-2}{5} + \frac{4}{5} = \frac{2}{5}$

5. If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b).

Sol. $a + ib = \left(\frac{1-i}{1+i}\right)^{10} = \left[\frac{(1-i)}{(1+i)} \cdot \frac{(1-i)}{(1-i)}\right]^{100} = \left[\frac{(1-i)^2}{1-i^2}\right]^{100}$

$$= \left(\frac{1-2i+i^2}{1+1}\right)^{100} = \left(\frac{-2i}{2}\right)^{100} = (i^4)^{25} = 1$$

$\therefore$ $(a, b) = (1, 0)$

Q6. If a = cos θ + i sin θ, then find the value of (1+a/1-a)

Sol: a = cos θ + i sin θ

$\therefore \quad \frac{1+a}{1-a} = \frac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta}$

$$= \frac{2\cos^2\frac{\theta}{2}+i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}-i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}-i\cos\frac{\theta}{2}\right)}$$

$$= \frac{i\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\left(i\sin\frac{\theta}{2}-i^2\cos\frac{\theta}{2}\right)} = \frac{i\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}{\sin\frac{\theta}{2}\left(i\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)} = i\cot\frac{\theta}{2}$$

7. If $(1+i)z = (1-i)\bar{z}$, then show that $z = -i\bar{z}$.

Sol. We have, $(1+i)z = (1-i)\bar{z}$

$\Rightarrow \quad z = \frac{1-i}{1+i}\bar{z} = \frac{(1-i)(1-i)}{(1+i)(1-i)}\bar{z} = \frac{(1-i)^2}{(1-i^2)}\bar{z} = \frac{1-2i+i^2}{1+1}\bar{z} = \frac{1-2i-1}{2}\bar{z} = -i\bar{z}$

8. If $z = x + iy$, then show that $z\bar{z} + 2(z+\bar{z}) + b = 0$, where $b \in R$, represents a circle.

Sol. Given that, $z = x + iy \quad \Rightarrow \quad \bar{z} = x - iy$

Now, $\quad z\bar{z} + 2(z+\bar{z}) + b = 0$

$\Rightarrow \quad (x+iy)(x-iy) + 2(x+iy+x-iy) + b = 0$

$\Rightarrow \quad x^2 + y^2 + 4x + b = 0$; this is the equation of a circle

9. If the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.

Sol. Let $z = x + iy$

Now, $\quad \frac{\bar{z}+2}{\bar{z}-1} = \frac{x-iy+2}{x-iy-1} = \frac{[(x+2)-iy][(x-1)+iy]}{[(x-1)-iy][(x-1)+iy]}$

$$= \frac{(x-1)(x+2)+y^2+i[(x+2)y-(x-1)y]}{(x-1)^2+y^2}$$

Given that real part is 4.

$\Rightarrow \quad \frac{(x-1)(x+2)+y^2}{(x-1)^2+y^2} = 4 \Rightarrow x^2+x-2+y^2 = 4(x^2-2x+1+y^2)$

$\Rightarrow \quad 3x^2 + 3y^2 - 9x + 6 = 0$, which represents a circle.

Hence, locus of z is circle

Q10. Show that the complex number z, satisfying the condition arg lies on arg (z-1/z+1) = π/4 lies on a circle.

Sol: Let z = x + iy

Given that, $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$

$\Rightarrow \quad \arg(z-1)-\arg(z+1)=\pi/4$

$\Rightarrow \quad \arg(x+iy-1)-\arg(x+iy+1)=\pi/4$

$\Rightarrow \quad \arg(x-1+iy)-\arg(x+1+iy)=\pi/4$

$\Rightarrow \quad \tan^{-1}\frac{y}{x-1}+\tan^{-1}\frac{y}{x+1}=\frac{\pi}{4} \Rightarrow \tan^{-1}\left[\frac{\frac{y}{x-1}-\frac{y}{x+1}}{1+\left(\frac{y}{x-1}\right)\left(\frac{y}{x+1}\right)}\right]=\frac{\pi}{4}$

$\Rightarrow \quad \frac{y(x+1-x+1)}{x^2-1+y^2}=\tan\frac{\pi}{4} \Rightarrow \frac{2y}{x^2+y^2-1}=1$

$\Rightarrow \quad x^2+y^2-1=2y$

$\Rightarrow \quad x^2+y^2-2y-1=0$, which represents a circle.

Q11. Solve the equation |z| = z + 1 + 2i.

Sol: We have |z| = z + 1 + 2i

Putting z = x + iy, we get

|x + iy| = x + iy + 1+2i

$\Rightarrow \quad \sqrt{x^2+y^2}=(x+1)+i(y+2) \qquad [\because |z|=\sqrt{x^2+y^2}]$

Comparing real and imaginary parts, we get

$\sqrt{x^2+y^2}=x+1;$

And $\quad 0=y+2 \Rightarrow y=-2$

Putting this value of y in $\sqrt{x^2+y^2}=x+1$, we get

$x^2+(-2)^2=(x+1)^2$

$\Rightarrow \quad x^2+4=x^2+2x+1 \qquad \Rightarrow x=3/2$

$\therefore \quad z=x+iy=3/2-2i$

Long Answer Type Questions

Q12. If |z + 1| = z + 2(1 + i), then find the value of z.

Sol: We have |z + 11 = z + 2(1+ i)

Putting z = x + iy, we get

Then, |x + iy + 11 = x + iy + 2(1 + i)

⇒|x + iy + l|=x + iy + 2(1 +i)

Comparing real and imaginary parts, we get

$$\sqrt{(x+1)^2 + y^2} = x + 2;$$

And $\quad y + 2 = 0 \Rightarrow y = -2$

Putting $y = -2$ into $\sqrt{(x+1)^2 + y^2} = x + 2$, we get

$$(x+1)^2 + (-2)^2 = (x+2)^2$$

$$\Rightarrow \quad x^2 + 2x + 1 + 4 = x^2 + 4x + 4 \quad \Rightarrow 2x = 1 \quad \Rightarrow x = \frac{1}{2}$$

$$\therefore \quad z = x + iy = \frac{1}{2} - 2i$$

Q13. If arg (z – 1) = arg (z + 3i), then find (x – 1) : y, where z = x + iy.

Sol: We have arg (z – 1) = arg (z + 3i), where z = x + iy

=> arg (x + iy – 1) = arg (x + iy + 3i)

=> arg (x – 1 + iy) = arg [x + i(y + 3)]

$$\Rightarrow \quad \tan^{-1}\frac{y}{x-1} = \tan^{-1}\frac{y+3}{x} \Rightarrow \frac{y}{x-1} = \frac{y+3}{x}$$

$$\Rightarrow \quad xy = (x-1)(y+3)$$

$$\Rightarrow \quad xy = xy - y + 3x - 3 \quad \Rightarrow 3x - 3 = y$$

$$\Rightarrow \quad \frac{x-1}{y} = \frac{1}{3}$$

$$\therefore \quad (x-1) : y = 1 : 3$$

Q14. Show that | z-2/z-3| = 2 represents a circle . Find its center and radius .

Sol: We have | z-2/z-3| = 2

Puttingz=x + iy, we get

$$\left|\frac{x+iy-2}{x+iy-3}\right|=2$$

$\Rightarrow$ $|x-2+iy|=2|x-3+iy| \Rightarrow \sqrt{(x-2)^2+y^2}=2\sqrt{(x-3)^2+y^2}$

$\Rightarrow$ $x^2-4x+4+y^2=4(x^2-6x+9+y^2) \Rightarrow 3x^2+3y^2-20x+32=0$

$\Rightarrow$ $x^2+y^2-\frac{20}{3}x+\frac{32}{3}=0 \Rightarrow \left(x-\frac{10}{3}\right)^2+y^2+\frac{32}{3}-\frac{100}{9}=0$

$\Rightarrow$ $\left(x-\frac{10}{3}\right)^2+(y-0)^2=\frac{4}{9}$

Hence, centre of the circle is $\left(\frac{10}{3},0\right)$ and radius is $\frac{2}{3}$.

Q15. If z-1/z+1 is a purely imaginary number (z ≠1), then find the value of |z|.

Sol: Let z = x + iy

$\Rightarrow$ $\frac{z-1}{z+1}=\frac{x+iy-1}{x+iy+1}, z\neq-1$

$$=\frac{x-1+iy}{x+1+iy}=\frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)}$$

$$=\frac{(x^2-1)+y^2+i[y(x+1)-y(x-1)]}{(x+1)^2+y^2}$$

It is given that $\frac{z-1}{z+1}$ is a purely imaginary.

$\Rightarrow$ $\frac{(x^2-1)+y^2}{(x+1)^2+y^2}=0 \quad \Rightarrow x^2-1+y^2=0 \quad \Rightarrow x^2+y^2=1$

$\Rightarrow$ $\sqrt{x^2+y^2}=1$

$\Rightarrow$ $|z|=1$

Alternative method:

Since $\dfrac{z-1}{z+1}$ is a purely imaginary number, we have

$$\frac{z-1}{z+1}+\overline{\left(\frac{z-1}{z+1}\right)}=0$$

$$\Rightarrow \quad \frac{z-1}{z+1}+\frac{\overline{z}-1}{\overline{z}+1}=0 \Rightarrow \frac{(z-1)(\overline{z}+1)+(z+1)(\overline{z}-1)}{(z+1)(\overline{z}+1)}=0$$

$$\Rightarrow \quad z\overline{z}+z-\overline{z}-1+z\overline{z}-z+\overline{z}-1=0 \Rightarrow 2z\overline{z}-2=0$$

$$\Rightarrow \quad |z|^2-1=0$$

$$\Rightarrow \quad |z|=1$$

16. z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\overline{z}_2$.

Sol. Let $z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$

Given that, $|z_1| = |z_2|$

And $\arg(z_1) + \arg(z_2) = \pi$

$\Rightarrow \quad \theta_1 + \theta_2 = \pi \quad \Rightarrow \theta_1 = \pi - \theta_2$

Now, $\quad z_1 = |z_2|[\cos(\pi-\theta_2) + i\sin(\pi-\theta_2)]$

$\Rightarrow \quad z_1 = |z_2|(-\cos\theta_2 + i\sin\theta_2) \Rightarrow z_1 = -|z_2|(\cos\theta_2 - i\sin\theta_2)$

$\Rightarrow \quad z_1 = -[|z_2|(\cos\theta_2 - i\sin\theta_2)] \quad \Rightarrow z_1 = -\overline{z}_2$

Q17. If $|z_1| = 1$ ($z_1 \neq -1$) and $z_2 = z_1 - 1 / z_1 + 1$, then show that real part of z_2 is zero .

Sol. Let $z_1 = x + iy$

$$\Rightarrow \quad |z_1| = \sqrt{x^2 + y^2} = 1 \text{ (given)}$$

Now, $$z_2 = \frac{z_1 - 1}{z_1 + 1}$$

$$= \frac{x + iy - 1}{x + iy + 1} = \frac{x - 1 + iy}{x + 1 + iy}$$

$$= \frac{(x - 1 + iy)(x + 1 - iy)}{(x + 1 + iy)(x + 1 - iy)}$$

$$= \frac{(x^2 - 1) + y^2 + i[y(x + 1) - y(x - 1)]}{(x + 1)^2 + y^2}$$

$$= \frac{x^2 + y^2 - 1 + 2iy}{(x + 1)^2 + y^2} = \frac{1 - 1 + 2iy}{(x + 1)^2 + y^2} \quad [\because x^2 + y^2 = 1]$$

$$= 0 + \frac{2yi}{(x + 1)^2 + y^2}$$

Hence, the real part of z_2 is zero.

Q18. If Z_1, Z_2 and Z_3, Z_4 are two pairs of conjugate complex numbers, then find arg (Z_1 / Z_4) + arg (Z_2 / Z_3)

Sol. It is given that z_1 and z_2 are conjugate complex numbers.

$\Rightarrow \quad z_2 = \bar{z}_1$

Also, z_3 and z_4 are conjugate complex numbers.

$\Rightarrow \quad z_4 = \bar{z}_3$

Now, $$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg\left(\frac{z_1}{z_4}\right)\left(\frac{z_2}{z_3}\right)$$

$$= \arg\left(\frac{z_1}{\bar{z}_3}\right)\left(\frac{\bar{z}_1}{z_3}\right) = \arg\left(\frac{z_1\bar{z}_1}{z_3\bar{z}_3}\right)$$

$$= \arg\left(\frac{|z_1|^2}{|z_3|^2}\right) = 0 \quad \left(\because \frac{|z_1|^2}{|z_3|^2} \text{ is purely real}\right)$$

19. If $|z_1| = |z_2| = \ldots = |z_n| = 1$, then show that $|z_1 + z_2 + z_3 + \ldots + z_n|$

$$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \cdots + \frac{1}{z_n}\right|$$

Sol. Given that $|z_1| = |z_2| = \ldots = |z_n| = 1$

$\Rightarrow \quad |z_1|^2 = |z_2|^2 = \ldots = |z_n|^2 = 1$

$\Rightarrow \quad z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = \ldots = z_n\bar{z}_n = 1$

$\Rightarrow \quad z_1 = \frac{1}{\bar{z}_1}, z_2 = \frac{1}{\bar{z}_2}, \ldots, z_n = \frac{1}{\bar{z}_n}$

Now, $|z_1 + z_2 + z_3 + z_4 + \ldots + z_n|$

$$= \left|\frac{z_1\bar{z}_1}{\bar{z}_1} + \frac{z_2\bar{z}_2}{z_2} + \frac{z_3\bar{z}_3}{\bar{z}_3} + \cdots + \frac{z_n\bar{z}_n}{z_n}\right| = \left|\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} + \cdots + \frac{1}{\bar{z}_n}\right|$$

$$= \left|\overline{\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \cdots + \frac{1}{z_n}}\right| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n}\right|$$

Q20. If for complex number z_1 and z_2, arg (z_1) − arg (z_2) = 0, then show that $|z_1 - z_2| = |z_1| - |z_2|$

Sol. Let $z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow$ $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

It is given that

$\arg(z_1) - \arg(z_2) = 0$

$\Rightarrow$ $\theta_1 - \theta_2 = 0 \quad \Rightarrow \theta_1 = \theta_2$

Now, $z_2 = |z_2|(\cos\theta_1 + i\sin\theta_1)$ $[\because \theta_1 = \theta_2]$

So, $z_1 - z_2 = (|z_1|\cos\theta_1 - |z_2|\cos\theta_1) + i(|z_1|\sin\theta_1 - |z_2|\sin\theta_1)$

$$\Rightarrow \quad |z_1 - z_2| = \sqrt{(|z_1|\cos\theta_1 - |z_2|\cos\theta_1)^2 + (|z_1|\sin\theta_1 - |z_2|\sin\theta_1)^2}$$

$$= \sqrt{|z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos^2\theta_1 - 2|z_1||z_2|\sin^2\theta_1}$$

$$= \sqrt{|z_1|^2 + |z_2|^2 - 2|z_1||z_2|[\cos^2\theta_1 + \sin^2\theta_1]}$$

$$= \sqrt{|z_1|^2 + |z_2|^2 - 2|z_1||z_2|} = \sqrt{(|z_1| - |z_2|)^2}$$

$$\Rightarrow \quad |z_1 - z_2| = |z_1| - |z_2|$$

Alternative method:

Let $A(z_1)$ and $B(z_2)$ be on the Argand plane.

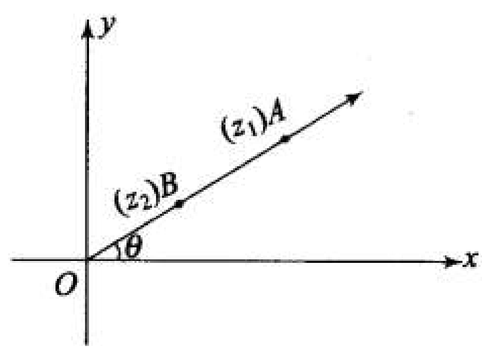
It is given that $\arg(z_1) = \arg(z_2)$.

So, A and B lie on the same ray emanating from origin O.

So, points O, A and B are collinear.

$\Rightarrow$ $AB = OA - OB$ (Assuming $|z_1| > |z_2|$)

$\Rightarrow$ $|z_1 - z_2| = |z_1| - |z_2|$



Q21. Solve the system of equations Re $(z^2) = 0$, $|z| = 2$.

Sol: Given that, $Re(z^2) = 0$, $|z| = 2$

Let $z = x + iy$. Then $|z| = \sqrt{x^2 + y^2}$.

Given that $\sqrt{x^2 + y^2} = 2$

$\Rightarrow \quad x^2 + y^2 = 4 \quad \ldots(i)$

Also, $z^2 = x^2 + 2ixy - y^2 = (x^2 - y^2) + 2ixy$

Now, $Re(z^2) = 0$

$\Rightarrow \quad x^2 - y^2 = 0 \quad (ii)$

Solving (i) and (ii), we get

$\Rightarrow \quad x^2 = y^2 = 2$

$\Rightarrow \quad x = \pm\sqrt{2}$ and $y = \pm\sqrt{2}$

$\therefore \quad z = x + iy = \pm\sqrt{2} \pm i\sqrt{2}$

Hence, we have four complex numbers.

Q22. Find the complex number satisfying the equation z + √2 |(z + 1)| + i = 0.

Sol. We have $z + \sqrt{2}\,|(z+1)| + i = 0 \quad \ldots(i)$

Putting $z = x + iy$, we get

$$x + iy + \sqrt{2}\,|x + iy + 1| + i = 0$$

$$\Rightarrow \quad x + i(1 + y) + \sqrt{2}\,[\sqrt{(x+1)^2 + y^2}] = 0$$

$$\Rightarrow \quad x + i(1 + y) + \sqrt{2}\sqrt{(x^2 + 2x + 1 + y^2)} = 0$$

Comparing real and imaginary parts to zero, we get

$$x + \sqrt{2}\sqrt{x^2 + 2x + 1 + y^2} = 0 \quad (ii)$$

And $y + 1 = 0 \Rightarrow y = -1$

Putting $y = -1$ into (ii), we get

$$x + \sqrt{2}\sqrt{x^2 + 2x + 1 + 1} = 0$$

$$\Rightarrow \quad \sqrt{2}\sqrt{x^2 + 2x + 2} = -x$$

$$\Rightarrow \quad 2x^2 + 4x + 4 = x^2 \quad \Rightarrow x^2 + 4x + 4 = 0 \Rightarrow (x + 2)^2 = 0$$

$$\Rightarrow \quad x = -2$$

$$\therefore \quad z = x + iy = -2 - i$$

23. Write the complex number $z = \dfrac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ in polar form.

Sol. $z = \dfrac{1-i}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$

$$= \frac{\sqrt{2}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}} = \frac{\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$

$$= \sqrt{2}\left[\cos\left(-\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{4} - \frac{\pi}{3}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right]$$

$$= -\sqrt{2}\left[\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right]$$

24. If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then show that $\bar{z}w = -i$.

Sol. Let $z = |z|(\cos\theta_1 + i\sin\theta_1)$ and $w = |w|(\cos\theta_2 + i\sin\theta_2)$

Given that, $|zw| = |z||w| = 1$

Also, $\arg(z) - \arg(w) = \frac{\pi}{2} \Rightarrow \theta_1 - \theta_2 = \frac{\pi}{2}$

Now, $\bar{z}w = |z|(\cos\theta_1 - i\sin\theta_1)\,|w|(\cos\theta_2 + i\sin\theta_2)$

$= |z||w|(\cos(-\theta_1) + i\sin(-\theta_1))(\cos\theta_2 + i\sin\theta_2)$

$= 1[\cos(\theta_2 - \theta_1) + i\sin(\theta_2 - \theta_1)] = [\cos(-\pi/2) + i\sin(-\pi/2)]$

$= 1[0 - i] = -i$

Fill in the blanks

25. Fill in the blanks of the following

(i) For any two complex numbers z_1, z_2 and any real numbers a, b,

$$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = \_\_\_\_.$$

(ii) The value of $\sqrt{-25} \times \sqrt{-9}$ is ____.

(iii) The number $\dfrac{(1-i)^3}{1-i^3}$ is equal to ____.

(iv) The sum of the series $i + i^2 + i^3 + \ldots$ upto 1000 terms is ____.

(v) Multiplicative inverse of $1 + i$ is ____.

(vi) If z_1 and z_2 are complex numbers such that $z_1 + z_2$ is a real number, then $z_1 =$ ____.

(vii) arg (z) + are $\bar{z}$ where, $(\bar{z} \neq 0)$ is ____.

(viii) If $|z + 4| \leq 3$, then the greatest and least values of $|z + 1|$ are ____ and ____.

(ix) If $\left|\dfrac{z-2}{z+2}\right| = \dfrac{\pi}{6}$, then the locus of z is ____.

(x) If $|z| = 4$ and arg $(z) = \dfrac{5\pi}{6}$, then $z =$ ____.

Sol. (i) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$$= |az_1|^2 + |bz_2|^2 - 2\text{Re}\,(az_1 \times b\bar{z}_2) + |bz_1|^2 + |az_2|^2 + 2\text{Re}(bz_1 \times a\bar{z}_2)$$

$$= (a^2 + b^2)|z_1|^2 + (a^2 + b^2)|z_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$$

(ii) $\sqrt{-25} \times \sqrt{-9} = i\sqrt{25} \times i\sqrt{9} = i^2(5 \times 3) = -15$

(iii) $\dfrac{(1-i)^3}{1-i^3} = \dfrac{(1-i)^3}{(1-i)(1+i+i^2)} = \dfrac{(1-i)^2}{i} = \dfrac{1+i^2-2i}{i} = \dfrac{-2i}{i} = -2$

(iv) $i + i^2 + i^3 + \ldots$ upto 1000 terms

$= (i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8) + \ldots$ 250 brackets

$= 0 + 0 + 0 \ldots + 0 \quad [\because i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$, where $n \in N]$

(v) Multiplicative inverse of $1 + i = \dfrac{1}{1+i} = \dfrac{1-i}{1-i^2} = \dfrac{1}{2}(1-i)$

(vi) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, which is real

$\Rightarrow y_1 + y_2 = 0 \quad \Rightarrow y_1 = -y_2$

$\because z_2 = x_1 - iy_1 \qquad$ [Assuming $x_1 = x_2$]

$\Rightarrow \qquad z_2 = \bar{z}_1$

(vii) arg (z) + arg $(\bar{z}) = \theta + (-\theta) = 0$

(viii) Given that, $|z + 4| \leq 3$

For the greatest value of $|z + 1|$,

$|z + 1| = |z + 4 - 3|$

$|z+1| = |z+4-3|$

or $\quad |z+1| \le |z+4| + |-3|$

or $\quad |z+1| \le 3+3$

or $\quad |z+1| \le 6$

So, greatest value of $|z+1|$ is 6.

We know that the least value of the modulus of a complex number is zero.

So, the least value of $|z+1|$ is zero.

(ix) We have $\left|\dfrac{z-2}{z+2}\right| = \dfrac{\pi}{6}$

$\Rightarrow \quad \dfrac{|x+iy-2|}{|x+iy+2|} = \dfrac{\pi}{6} \Rightarrow \dfrac{|x-2+iy|}{|x+2+iy|} = \dfrac{\pi}{6}$

$\Rightarrow \quad 6|x-2+iy| = \pi|x+2+iy| \Rightarrow 36|x-2+iy|^2 = \pi^2|x+2+iy|^2$

$\Rightarrow \quad 36[x^2 - 4x + 4 + y^2] = \pi^2[x^2 + 4x + 4 + y^2]$

$\Rightarrow \quad (36-\pi^2)x^2 + (36-\pi^2)y^2 - (144+4\pi^2)x + 144 - 4\pi^2 = 0$, which is a circle.

(x) Let $z = |z|(\cos\theta + i\sin\theta)$

Where $\theta = \arg(z)$

Given that $|z| = 4$ and $\arg(z) = \dfrac{5\pi}{6}$

$\Rightarrow \quad z = 4\left[\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right]$ (z lies in II quadrant)

$= 4\left[-\dfrac{\sqrt{3}}{2} + i\dfrac{1}{2}\right] = -2\sqrt{3} + 2i$

True/False Type Questions

Q26. State true or false for the following.

(i) The order relation is defined on the set of complex numbers.

(ii) Multiplication of a non-zero complex number by -i rotates the point about origin through a right angle in the anti-clockwise direction.

(iii) For any complex number z, the minimum value of |z| + |z – 11 is 1.

(iv) The locus represented by |z — 11= |z — i| is a line perpendicular to the join of the points (1,0) and (0, 1).

(v) If z is a complex number such that z ≠ 0 and Re(z) = 0, then Im (z^2) = 0.

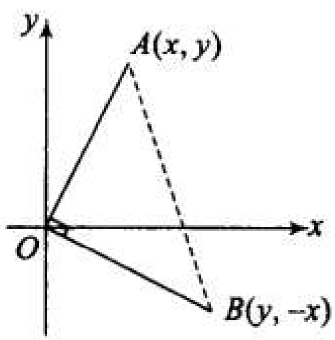
(vi) The inequality |z – 4| < |z – 2| represents the region given by x > 3.

(vii) Let Z_1 and Z_2 be two complex numbers such that |z, + z_2| = |z_1 j + |z_2|, then arg (z_1 – z_2) = 0.

(viii) 2 is not a complex number.

Sol:(i) False
We can compare two complex numbers when they are purely real. Otherwise comparison of complex numbers is not possible or has no meaning.

(ii) False
Let $z = x + iy$, where $x, y > 0$
i.e., z or point A(x, y) lies in first quadrant. Now, $-iz = -i(x + iy)$
$= -ix - i^2y = y - ix$
Now, point B(y, – x) lies in fourth quadrant. Also, ∠AOB = 90°
Thus, B is obtained by rotating A in clockwise direction about origin.



(iii) **True**

$$|z| + |z - 1|$$

We know that $|z_1| + |z_2| \geq |z_1 - z_1|$

$\Rightarrow \quad |z| + |z - 1| \geq |z - (z - 1)| \quad \Rightarrow |z| + |z - 1| \geq 1$

So, minimum value of $|z| + |z - 1|$ is 1.

Alternative method:

Let $A(z)$ and $B(1)$.

$\Rightarrow |z| + |z - 1| = OA + AB$, where O is origin

From triangular inequality, we get

$$OA + AB \geq OB$$

$\Rightarrow \quad (OA + AB)_{\text{min.}} = OB = 1$

(iv) **True**

We have, $|z - 1| = |z - i|$

Putting $z = x + iy$, we get

$\Rightarrow \quad |x - 1 + iy| = |x - i(1 - y)|$

$\Rightarrow \quad (x - 1)^2 + y^2 = x^2 + (1 - y)^2 \Rightarrow x^2 - 2x + 1 + y^2 = x^2 + 1 + y^2 - 2y$

$\Rightarrow \quad -2x + 1 = 1 - 2y \qquad \Rightarrow -2x + 2y = 0 \qquad \Rightarrow x - y = 0$

Now, equation of a line through the points (1, 0) and (0, 1) is:

$$y - 0 = \frac{1 - 0}{0 - 1}(x - 1)$$

or $\quad x + y = 1$

This line is perpendicular to the line $x - y = 0$.

(v) **False**

Let $z = x + iy$, $z \neq 0$ and $\text{Re}(z) = 0$.

i.e., $\quad x = 0$

$\therefore \quad z = iy$

$$\text{Im}(z^2) = i^2 y^2 = -y^2 \neq 0$$

(vi) **True**

We have, $|z - 4| < |z - 2|$

Putting $z = x + iy$, we get

$$|x - 4 + iy| < |x - 2 + iy|$$

$\Rightarrow \quad \sqrt{(x - 4)^2 + y^2} < \sqrt{(x - 2)^2 + y^2}$

$\Rightarrow \quad (x-4)^2 + y^2 < (x-2)^2 + y^2$

$\Rightarrow \quad x^2 - 8x + 16 + y^2 < x^2 - 4x + 4 + y^2$

$\Rightarrow \quad -8x + 16 < -4x + 4$

$\Rightarrow \quad 4x > 12$

$\Rightarrow \quad x > 3$

(vii) **False**

$|z_1 + z_2| = |z_1| + |z_2|$

$\Rightarrow \quad |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2) = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow \quad 2\text{Re}(z_1\bar{z}_2) = 2|z_1||z_2| \qquad \Rightarrow \cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \quad \theta_1 - \theta_2 = 0 \qquad \Rightarrow \arg(z_1) - \arg(z_2) = 0$

(viii) **False**

We know that, any real number is also a complex number.

Matching Column Type Questions

Q24. Match the statements of Column A and Column B.

Column A			Column B
(a)	The polar form of i + √3 is	(i)	Perpendicular bisector of segment joining (-2, 0) and (2,0)
(b)	The amplitude of- 1 + √-3 is	(ii)	On or outside the circle having centre at (0, -4) and radius 3.
(c)	It |z + 2| = |z – 2|, then locus of z is	(iii)	2/3
(d)	It |z + 2i| = |z – 2i|, then locus of z is	(iv)	Perpendicular bisector of segment joining (0, -2) and (0,2)
(e)	Region represented by |z + 4i| ≥ 3 is	(v)	2(cos /6 +I sin /6)
(0	Region represented by |z + 4| ≤ 3 is	(Vi)	On or inside the circle having centre (-4,0) and radius 3 units.
(g)	Conjugate of 1+2i/1-I lies in	(vii)	First quadrant

(h)	Reciprocal of 1 – i lies in	(viii)	Third quadrant

Sol. (a) Given that, $z = i + \sqrt{3}$

So, $|z| = |i + \sqrt{3}| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

Also, z lies in first quadrant.

$\Rightarrow \arg(z) = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

So, the polar from of z is $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

(b) We have, $z = -1 + \sqrt{-3} = -1 + i\sqrt{3}$

Here z lies in second quadrant.

$\Rightarrow \quad \arg(z) = \text{amp}(z) = \pi - \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \pi - \tan^{-1}\sqrt{3} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

(c) Given that, $|z + 2| = |z - 2|$

$\Rightarrow \quad |x + 2 + iy| = |x - 2 + iy|$

$\Rightarrow \quad (x+2)^2 + y^2 = (x-2)^2 + y^2 \quad \Rightarrow x^2 + 4x + 4 = x^2 - 4x + 4$

$\Rightarrow \quad 8x = 0 \qquad \therefore\ x = 0$

It is a straight line which is a perpendicular bisector of segment joining the points (–2, 0) and (2, 0).

(d) We have $|z + 2i| = |z - 2i|$

Putting $z = x + iy$, we get

$\Rightarrow \quad |x + i(y+2)|^2 = |x + i(y-2)|^2 \qquad \Rightarrow x^2 + (y+2)^2 = x^2 + (y-2)^2$

$\Rightarrow \quad 4y = 0 \qquad \Rightarrow y = 0$

It is a straight line, which is a perpendicular bisector of segment joining (0, –2) and (0, 2).

Alternative method:

We know that $|z_1 - z_2|$ = distance between z_1 and z_2

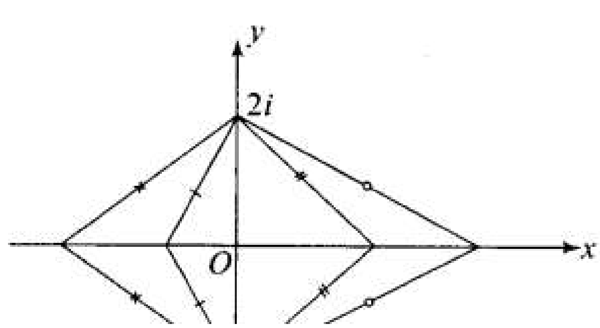
Now, $|z + 2i| = |z - 2i|$

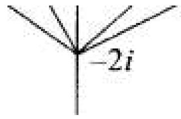
$\Rightarrow \quad |z - (-2i)| = |z - 2i|$

$\Rightarrow$ Distance between z and $-2i$ = Distance between z and $2i$

Thus, z lies on the perpendicular bisector of the line segment joining $-2i$ and $2i$.

Hence, z lies on the x-axis as shown in the figure.





(e) Given that, $|z + 4i| \geq 3$

$\Rightarrow \quad |x + iy + 4i| \geq 3 \qquad \Rightarrow |x + i(y + 3)| \geq 3$

$\Rightarrow \quad \sqrt{x^2 + (y+4)^2} \geq 3 \qquad \Rightarrow x^2 + y^2 + 8y + 16 \geq 9$

$\Rightarrow \quad x^2 + y^2 + 8y + 7 \geq 0$

This represents the region on or outside the circle having centre (0, –4) and radius 3.

(f) Given that, $|z + 4| \leq 3$

$\Rightarrow \quad |x + iy + 4| \leq 3 \qquad \Rightarrow |x + 4 + iy| \leq 3$

$\Rightarrow \quad \sqrt{(x+4)^2 + y^2} \leq 3 \qquad \Rightarrow (x + 4)^2 + y^2 \leq 9$

$\Rightarrow \quad x^2 + 8x + 16 + y^2 \leq 9 \qquad \Rightarrow x^2 + 8x + y^2 + 7 \leq 0$

This represents the region on or inside circle having centre (–4, 0) and radius 3.

(g) $z = \dfrac{1+2i}{1-i} = \dfrac{(1+2i)(1+i)}{(1+i)(1+i)} = \dfrac{1+2i+i+2i^2}{1-i^2}$

$$= \frac{1-2+3i}{1+1} = \frac{-1}{2} + \frac{3i}{2}$$

Hence, $\bar{z}$ lies in the third quadrant.

(h) Given that, $z = 1 - i$

$\Rightarrow \quad \dfrac{1}{z} = \dfrac{1}{1-i} = \dfrac{1+i}{(1-i)(1+i)} = \dfrac{1+i}{1-i^2} = \dfrac{1}{2}(1+i)$

Thus, reciprocal of z lies in first quadrant.

Q28. What is the conjugate of 2-i / $(1 - 2i)^2$

Sol. We have $z = \frac{2-i}{(1-2i)^2}$

$$\Rightarrow \quad z = \frac{2-i}{1+4i^2-4i} = \frac{2-i}{1-4-4i} = \frac{2-i}{-3-4i}$$

$$= \frac{(2-i)}{-(3+4i)} = -\left[\frac{(2-i)(3-4i)}{(3+4i)(3-4i)}\right]$$

$$= -\left(\frac{6-8i-3i+4i^2}{9+16}\right) = -\frac{(-11i+2)}{25}$$

$$= \frac{-1}{25}(2-11i) = \frac{1}{25}(-2+11i)$$

$$\therefore \quad \bar{z} = \frac{1}{25}(-2-11i) = \frac{-2}{25} - \frac{11}{25}i$$

Q29. If $|Z_1| = |Z_2|$, is it necessary that $Z_1 = Z_2$?

Sol: If $|Z_1| = |Z_2|$ then z_1 and z_2 are at the same distance from origin.

But if $\arg(Z_1) \neq \arg(z_2)$, then z_1 and z_2 are different.

So, if $(z_1| = |z_2|$, then it is not necessary that $z_1 = z_2$.

Consider $Z_1 = 3 + 4i$ and $Z_2 = 4 + 3i$

Q30.If $(a^2+1)^2 / 2a - i = x + iy$, then what is the value of $x^2 + y^2$?

Sol: $(a^2+1)^2 / 2a - i = x + iy$

$$\Rightarrow \quad \left|\frac{(a^2+1)^2}{2a-i}\right| = |x+iy|$$

$$\Rightarrow \quad \frac{|(a^2+1)^2|}{|2a-i|} = |x+iy| \Rightarrow \frac{(a^2+1)^2}{\sqrt{(2a)^2+(-1)^2}} = \sqrt{x^2+y^2}$$

$$\therefore \quad x^2+y^2 = \frac{(a^2+1)^4}{4a^2+1}$$

Q31. Find the value of z, if |z| = 4 and arg (z) = 5π/6

Sol. Let $z = |z| (\cos \theta + i \sin \theta)$, where $\theta = \arg(z)$.

Given that, $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$.

$$\Rightarrow \quad z = 4\left[\cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6}\right] \quad (z \text{ lies in II quadrant})$$

$$= 4\left[-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right] = -2\sqrt{3} + 2i$$

32. Find the value of $\left|(1+i)\frac{(2+i)}{(3+i)}\right|$.

Sol. $$\left|(1+i)\frac{(2+i)}{(3+i)}\right| = |1+i|\frac{|2+i|}{|3+i|} = \sqrt{1^2+1^2}\,\frac{\sqrt{2^2+1^2}}{\sqrt{3^2+1^2}} = \sqrt{2}\,\frac{\sqrt{5}}{\sqrt{10}} = 1$$

33. Find the principal argument of $(1 + i\sqrt{3})^2$.

Sol. We have,

$$z = (1+i\sqrt{3})^2 = 1 - 3 + 2i\sqrt{3} = -2 + i(2\sqrt{3})$$

So, z lies in second quadrant.

$$\Rightarrow \quad \arg(z) = \pi - \tan^{-1}\left|\frac{2\sqrt{3}}{-2}\right| = \pi - \tan^{-1}\sqrt{3} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

Q34. Where does z lies, if | z – 5i / z + 5i | = 1?

Sol: We have | z – 5i / z + 5i |

$\Rightarrow \quad |z - 5i| = |z + 5i| \quad \Rightarrow |x + iy - 5i| = |x + iy + 5i|$

$\Rightarrow \quad |x + i(y - 5)|^2 = |x + i(y + 5)|^2 \quad \Rightarrow x^2 + (y - 5)^2 = x^2 + (y + 5)^2$

$\Rightarrow \quad 20y = 0 \quad \Rightarrow y = 0$

So, z lies on the x-axis (real axis).

Alternative method:

We know that $|z_1 - z_2|$ = Distance between z_1 and z_2

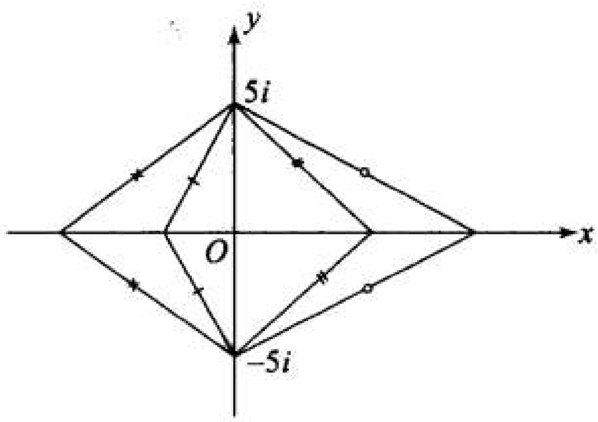
Now, $\left|\dfrac{z - 5i}{z + 5i}\right| = 1$

$\Rightarrow \quad |z - 5i| = |z + 5i| \quad \Rightarrow |z - 5i| = |z - (-5i)|$

$\Rightarrow$ Distance between 'z' and '$5i$' = Distance between 'z' and '$-5i$'

This means that z lies on the perpendicular bisector of the line segment joining '$5i$' and '$-5i$'.

Hence, z lies on the x-axis as shown in the figure.



Instruction for Exercises 35-40: Choose the correct answer from the given four options indicated against each of the Exercises.

Q35. sin x + i cos 2x and cos x – i sin 2x are conjugate to each other for

(a) $x = n\pi$ (b) $x = \left(n + \frac{1}{2}\right)\frac{\pi}{2}$

(c) $x = 0$ (d) no value of x

Sol. **(d)** Given that,

$\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other

$\Rightarrow \quad \overline{\sin x + i\cos 2x} = \cos x - i\sin 2x$

$\Rightarrow \quad \sin x - i\cos 2x = \cos x - i\sin 2x$

On comparing real and imaginary parts of both the sides, we get

$\sin x = \cos x$ and $\cos 2x = \sin 2x$

$\Rightarrow \quad \tan x = 1$ and $\tan 2x = 1$

Now, $\tan 2x = 1$

$\Rightarrow \frac{2\tan x}{1 - \tan^2 x} = 1$, which is not satisfied by $\tan x = 1$

Hence, no value of x is possible.

36. The real value of α for which the expression $\frac{1 - i\sin\alpha}{1 + 2i\sin\alpha}$ is purely real is

(a) $(n+1)\frac{\pi}{2}$ (b) $(2n+1)\frac{\pi}{2}$

(c) $n\pi$ (d) none of these

Sol. **(c)** $z = \frac{1 - i\sin\alpha}{1 + 2i\sin\alpha}$

$$= \frac{(1 - i\sin\alpha)(1 - 2i\sin\alpha)}{(1 + 2i\sin\alpha)(1 - 2i\sin\alpha)} = \frac{1 - i\sin\alpha - 2i\sin\alpha + 2i^2\sin^2\alpha}{1 - 4i^2\sin^2\alpha}$$

$$= \frac{1 - 3i\sin\alpha - 2\sin^2\alpha}{1 + 4\sin^2\alpha} = \frac{1 - 2\sin^2\alpha}{1 + 4\sin^2\alpha} - \frac{3i\sin\alpha}{1 + 4\sin^2\alpha}$$

It is given that z is a purely real.

$\Rightarrow \quad \frac{-3\sin\alpha}{1 + 4\sin^2\alpha} = 0 \quad \Rightarrow -3\sin\alpha = 0 \quad \Rightarrow \sin\alpha = 0$

$\Rightarrow \quad \alpha = n\pi, n \in I$

37. If $z = x + iy$ lies in the third quadrant, then $\frac{\bar{z}}{z}$ also lies in the third quadrant, if

(a) $x > y > 0$ (b) $x < y < 0$

(c) $y < x < 0$ (d) $y > x > 0$

Sol. **(c)** Since $z = x + iy$ lies in the third quadrant, we get

$$x < 0 \text{ and } y < 0$$

Now, $\frac{\bar{z}}{z} = \frac{x - iy}{x + iy} = \frac{(x - iy)(x - iy)}{(x + iy)(x - iy)} = \frac{x^2 - y^2 - 2ixy}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2} - \frac{2ixy}{x^2 + y^2}$

Since $\frac{\bar{z}}{z}$ also lies in third quadrant, we get

$$\frac{x^2 - y^2}{x^2 + y^2} < 0 \text{ and } \frac{-2xy}{x^2 + y^2} < 0$$

$\Rightarrow$ $x^2 - y^2 < 0$ and $-2xy < 0$

$\Rightarrow$ $x^2 < y^2$ and $xy > 0$

But $x, y < 0$

$\Rightarrow$ $y < x < 0$

38. The value of $(z+3)(\overline{z}+3)$ is equivalent to

(a) $|z+3|^2$ (b) $|z-3|$ (c) z^2+3 (d) none of these

Sol. **(a)** Let $z = x + iy$. Then

$$(z+3)(\overline{z}+3) = (x+iy+3)(x-iy+3)$$
$$= (x+3)^2 - (iy)^2 = (x+3)^2 + y^2 = |x+3+iy|^2 = |z+3|^2$$

Alternative method:

$$(z+3)(\overline{z}+3) = (z+3)\overline{(z+3)}$$
$$= |z+3|^2 \qquad (\because z\overline{z} = |z|^2)$$

39. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then

(a) $x = 2n+1$ (b) $x = 4n$

(c) $x = 2n$ (d) $x = 4n+1$

where, $n \in N$

Sol. **(b)** $\left(\frac{1+i}{1-i}\right)^x = 1$

$$\Rightarrow \left[\frac{(1+i)(1+i)}{(1-i)(1+i)}\right]^x = 1 \Rightarrow \left[\frac{1+2i+i^2}{1-i^2}\right]^x = 1 \Rightarrow \left[\frac{2i}{1+1}\right]^x = 1$$

$$\Rightarrow i^x = 1$$

$$\Rightarrow x = 4n, n \in N$$

Q41. Which of the following is correct for any two complex numbers z_1 and z_2?

(a) $|z_1 z_2| = |z_1||z_2|$ (b) $\arg(z_1 z_2) = \arg(z_1) \times \arg(z_2)$

(c) $|z_1 + z_2| = |z_1| + |z_2|$ (d) $|z_1 + z_2| \geq |z_1| - |z_2|$

Sol. **(a)** Clearly, $|z_1 z_2| = |z_1||z_2|$

Proof:

Let $z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$

Now,
$$\begin{aligned} z_1 z_2 &= |z_1||z_2|(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= |z_1||z_2|[\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 \\ &\quad + i^2\sin\theta_1\sin\theta_2] \\ &= |z_1||z_2|[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \end{aligned}$$

$\Rightarrow$ $|z_1 z_2| = |z_1||z_2|$

And $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

$|z_1 + z_2| = |z_1| + |z_2|$ is true only when z_1, z_2 and O(origin) are collinear.

Also, $|z_1 + z_2| \geq ||z_1| - |z_2||$