Short Answer Type Questions

Q1. Give an example of a statement P(n) which is true for all n≥ 4 but P(l), P(2) and P(3) are not true. Justify your answer.

Sol. Consider the statement P(n): 3n < n!

For n = 1, 3 x 1 < 1!, which is not true
For n = 2, 3 x 2 < 2!, which is not true
For n = 3, 3 x 3 < 3!, which is not true
For n = 4, 3 x 4 < 4!, which is true
For n = 5, 3 x 5 < 5!, which is true

Q2. Give an example of a statement P(n) which is true for all Justify your answer.

Sol. Consider the statement:

$$P(n): 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

For $n = 1$, $1^3 = \dfrac{1^2(1+1)^2}{4} = 1$

Thus, $P(1)$ is true.

For $n = 2$, $1^3 + 2^3 = 1 + 8 = 9$ and $\dfrac{2^2(2+1)^2}{4} = 9$

Thus, $P(2)$ is true.

For $n = 3$, $1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$ and $\dfrac{3^2(3+1)^2}{4} = 36$

Thus, $P(3)$ is true.

Hence, the given statement is true for all n.

Instruction for Exercises 3-16: Prove each of the statements in these Exercises by the Principle of Mathematical Induction.

Q3. 4^n – 1 is divisible by 3, for each natural number

Sol: Let P(n): $4^n - 1$ is divisible by 3 for each natural number n.

Now, P(l): $4^1 - 1 = 3$, which is divisible by 3 Hence, P(l) is true.

Let us assume that P(n) is true for some natural number n = k.

P(k): $4^k - 1$ is divisible by 3

or $4^k - 1 = 3m, m \in N$ (i)

Now, we have to prove that P(k + 1) is true.

P(k+ 1): $4^{k+1} - 1$

$= 4^k \cdot 4 - l$

= 4(3m + 1) – 1 [Using (i)]

= 12 m + 3

= 3(4m + 1), which is divisible by 3 Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction P(n) is true for all natural numbers n.

Q4. 2^{3n} – 1 is divisible by 7, for all natural numbers

Sol: Let P(n): $2^{3n} - 1$ is divisible by 7

Now, P(1): $2^3 - 1 = 7$, which is divisible by 7.

Hence, P(l) is true.

Let us assume that P(n) is true for some natural number n = k.

P(k): $2^{3k} - 1$ is divisible by 7.

or $2^{3k} - 1 = 7m, m \in N$ (i)

Now, we have to prove that P(k + 1) is true.

P(k+ 1): $2^{3(k+1)} - 1$

$= 2^{3k} \cdot 2^3 - 1$

= 8(7 m + 1) – 1

= 56 m + 7

= 7(8m + 1), which is divisible by 7.

Thus, P(k + 1) is true whenever P(k) is true.

So, by the principle of mathematical induction P(n) is true for all natural numbers n.

Q5. n^3 – 7n + 3 is divisible by 3, for all natural numbers

Sol: Let P(n): $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n.

Now P(l): $(l)^3 - 7(1) + 3 = -3$, which is divisible by 3.

Hence, P(l) is true.

Let us assume that P(n) is true for some natural number n = k.

P(k) = $K^3 - 7k + 3$ is divisible by 3

or $K^3 - 7k + 3 = 3m, m \in N$ (i)

Now, we have to prove that P(k + 1) is true.

P(k+ 1):$(k + l)^3 - 7(k + 1) + 3$

$= k^3 + 1 + 3k(k + 1) - 7k - 7 + 3 = k^3 - 7k + 3 + 3k(k + l) - 6$

$= 3m + 3[k(k+l)-2]$ [Using (i)]

$= 3[m + (k(k + 1) - 2)]$, which is divisible by 3 Thus, P(k + 1) is true whenever P(k) is true.

So, by the principle of mathematical induction P(n) is true for all natural numbers n.

Q6. $3^{2n} - 1$ is divisible by 8, for all natural numbers

Sol: Let P(n): $3^{2n} - 1$ is divisible by 8, for all natural numbers n.

Now, P(l): $3^2 - 1 = 8$, which is divisible by 8.

Hence, P(l) is true.

Let us assume that, P(n) is true for some natural number n = k.

P(k): $3^{2k} - 1$ is divisible by 8

or $3^{2k} - 1 = 8m, m \in N$ (i)

Now, we have to prove that P(k + 1) is true.

P(k+ 1): $3^{2(k+1)} - l$

$= 3^{2k} \cdot 3^2 - 1$

$= 9(8m + 1) - 1$ (using (i))

$= 72m + 9 - 1$

$= 72m + 8$

$= 8(9m + 1)$, which is divisible by 8 Thus P(k + 1) is true whenever P(k) is true.

So, by the principle of mathematical induction P(n) is true for all natural numbers n.

Q7. For any natural number n, $7^n - 2^n$ is divisible by 5.

Sol: Let P(n): $7^n - 2^n$ is divisible by 5, for any natural number n.

Now, P(l) = $7^1 - 2^1 = 5$, which is divisible by 5.

Hence, P(l) is true.

Let us assume that, P(n) is true for some natural number n = k.

∴ P(k) = $7^k - 2^k$ is divisible by 5

or $7^k - 2^k = 5m, m \in N$ (i)

Now, we have to prove that P(k + 1) is true.

P(k+ 1): $7^{k+1} - 2^{k+1}$

$= 7^k \cdot 7 - 2^k \cdot 2$

$= (5 + 2)7^k - 2^k \cdot 2$

$= 5.7^k + 2.7^k - 2 \cdot 2^k$

$= 5.7^k + 2(7^k - 2^k)$

$= 5 \cdot 7^k + 2(5m)$ (using (i))

$= 5(7^k + 2m)$, which divisible by 5.

Thus, P(k + 1) is true whenever P(k) is true.

So, by the principle of mathematical induction P(n) is true for all natural numbers n.

Q8. For any natural number n, $x^n - y^n$ is divisible by x -y, where x and y are any integers with x ≠y

Sol: Let P(n) : $x^n – y^n$ is divisible by x – y, where x and y are any integers with x≠y.

Now, P(l): $x^1 - y^1$ = x-y, which is divisible by (x-y)

Hence, P(l) is true.

Let us assume that, P(n) is true for some natural number n = k.

P(k): $x^k - y^k$ is divisible by (x – y)

or $x^k - y^k = m(x-y), m \in N$ ...(i)

Now, we have to prove that P(k + 1) is true.

P(k+l): $x^{k+l} - y^{k+l}$

$= x^k \cdot x - x^k \cdot y + x^k \cdot y - y^k y$

$= x^k(x-y) + y(x^k - y^k)$

$= x^k(x – y) + ym(x – y)$ (using (i))

$= (x - y) [x^k + ym]$, which is divisible by (x-y)

Hence, P(k + 1) is true whenever P(k) is true.

So, by the principle of mathematical induction P(n) is true for any natural number n.

Q9. n^3 -n is divisible by 6, for each natural number n≥

Sol: Let P(n): $n^3 – n$ is divisible by 6, for each natural number n> 2.

Now, P(2): $(2)^3 - 2 = 6$, which is divisible by 6.

Hence, P(2) is true.

Let us assume that, P(n) is true for some natural number n = k.

P(k): $k^3 – k$ is divisible by 6

or $k^3 - k = 6m, m \in N$ (i)

Now, we have to prove that P(k + 1) is true.

P(k+ 1): $(k+ l)^3 - (k+ 1)$

$= k^3 + 1 + 3k(k+ l) - (k+ 1)$

$= k^3 + 1 + 3k^2 + 3k - k - 1 = (k^3 - k) + 3k(k+ 1)$

$= 6m + 3 k(k +1)$ (using (i))

Above is divisible by 6. (∴ k(k + 1) is even)

Hence, P(k + 1) is true whenever P(k) is true.

So, by the principle of mathematical induction P(n) is true for any natural number n,n≥ 2.

Q10. $n(n^2 + 5)$ is divisible by 6, for each natural number

Sol: Let P(n): $n(n^2 + 5)$ is divisible by 6, for each natural number.

Now P(l): $1 (l^2 + 5) = 6$, which is divisible by 6.

Hence, P(l) is true.

Let us assume that P(n) is true for some natural number n = k.

P(k): $k(k^2 + 5)$ is divisible by 6.

or $K (k^2+ 5) = 6m, m\in N$ (i)

Now, we have to prove that P(k + 1) is true.

$P(K+l):(K+l)[(K+l)^2 + 5]$

$= (K + l)[K^2 + 2K+6]$

$= K^3 + 3 K^2 + 8K + 6$

$= (K^2 + 5K) + 3 K^2 + 3K + 6 =K(K^2 + 5) + 3(K^2 + K + 2)$

$= (6m) + 3(K^2 + K + 2)$ (using (i))

Now, $K^2 + K + 2$ is always even if A is odd or even.

So, $3(K^2 + K + 2)$ is divisible by 6 and hence, $(6m) + 3(K^2 + K + 2)$ is divisible by 6.

Hence, P(k + 1) is true whenever P(k) is true.

So, by the principle of mathematical induction P(n) is true for any natural number n.

Q11. $n^2 < 2^n$, for all natural numbers n ≥

Sol: Let P(n): $n^2 < 2^n$ for all natural numbers $n\geq 5$.

Now P(5): $5^2 < 2^5$ or 25 < 32, which is true.

Hence, P(5) is true.

Let us assume that P(n) is true for some natural number n = k.

∴ P(k): $k^2 < 2^k$ (i)

Now, to prove that P(k + 1) is true, we have to show that P(k+ 1): $(k+ l)^2 <2^{k+1}$

Using (i), we get

$(k + l)^2 = k^2 + 2k + 1 < 2^k + 2k + 1$ (ii)

Now let, $2^k + 2k + 1 < 2^{k+1}$ (iii)

$\therefore 2^k + 2k + 1 < 2 \cdot 2^k$

$2k + 1 < 2^k$, which is true for all k > 5 Using (ii) and (iii), we get $(k + l)^2 < 2^{k+1}$ Hence, P(k + 1) is true whenever P(k) is true.

So, by the principle of mathematical induction P(n) is true for any natural number $n, n\geq 5$.

Q12. 2n<(n + 2)! for all natural numbers

Sol: Let P(n): 2n < (n + 2)! for all natural numbers n.

P(1): 2 < (1 + 2)! or 2 < 3! or 2 < 6, which is true.

Hence,P(l) is true.

Let us assume that P(n) is true for some natural number n = k.

$P(k) : 2k < (k + 2)!$ (i)

To prove that P(k + 1) is true, we have to show that

$P(k + 1): 2(k+ 1) < (k + 1 + 2)!$

or $2(k+ 1) < (k + 3)!$

Using (i), we get

$2(k + 1) = 2k + 2 < (k+2)! + 2$ (ii)

Now let, $(k + 2)! + 2 < (k + 3)!$ (iii)

=> $2 < (k+ 3)! - (k+2)!$

=> $2 < (k + 2)! [k+ 3-1]$

=> $2 < (k+ 2)! (k + 2)$, which is true for any natural number.

Using (ii) and (iii), we get $2(k + 1) < (k + 3)!$

Hence, P(k + 1) is true whenever P(k) is true.

So, by the principle of mathematical induction P(n) is true for any natural number n.

13. $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.

Sol. Let $P(n)$: $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$, for all natural numbers $n \geq 2$.

$P(2)$: $\sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$, which is true.

Hence, $P(2)$ is true.

Let us assume that $P(n)$ is true for some natural number $n = k$

$\therefore \quad P(k): \sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} \quad$ (i)

To prove that $P(k+1)$ is true, we have to show that

$$P(k+1): \sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

Now, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$> \sqrt{k} + \frac{1}{\sqrt{k+1}}$ [Using (i)]

$> \sqrt{k+1}$ $\left(\because \frac{1}{\sqrt{k+1}} > 0\right)$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for any natural number n, $n \geq 2$.

Q14. 2 + 4 + 6+… + 2n = n^2 + n, for all natural numbers

Sol: Let P(n) :2 + 4 + 6+ …+2 n = $n^2 + n$

P(l): 2 = $l^2 + 1$ = 2, which is true

Hence, P(l) is true.

Let us assume that P(n) is true for some natural number n = k.

∴ P(k): 2 + 4 + 6 + .,.+2k = $k^2 + k$ (i)

Now, we have to prove that P(k + 1) is true.

P(k + l):2 + 4 + 6 + 8+ …+2k+ 2 (k +1)

= k^2 + k + 2(k+ 1) [Using (i)]

$= k^2 + k + 2k + 2$

$= k^2 + 2k+1+k+1$

$= (k + 1)^2 + k+ 1$

Hence, P(k + 1) is true whenever P(k) is true.

So, by the principle of mathematical induction P(n) is true for any natural number n.

Q15. $1 + 2 + 2^2 + … + 2^n = 2^{n+1} – 1$ for all natural numbers

Sol: Let P(n): $1 + 2 + 2^2 + … + 2^n = 2^{n+1} – 1$, for all natural numbers n

P(1): $1 =2^{0+1} — 1 = 2 — 1 = 1$, which is true.

Hence, ,P(1) is true.

Let us assume that P(n) is true for some natural number n = k.

P(k): $l+2 + 2^2+…+2^k = 2^{k+1}-l$ (i)

Now, we have to prove that P(k + 1) is true.

P(k+1): $1+2 + 2^2+ …+2^k + 2^{k+1}$

$= 2^{k+1} – 1 + 2^{k+1}$ [Using (i)]

$= 2.2^{k+l}– 1 = 1$

$= 2^{(k+1)+1}-1$

Hence, P(k + 1) is true whenever P(k) is true.

So, by the principle of mathematical induction P(n) is true for any natural number n.

Q16. 1 + 5 + 9 + … + (4n – 3) = n(2n – 1), for all natural numbers

Sol: Let P(n): 1 + 5 + 9 + … + (4n – 3) = n(2n – 1), for all natural numbers n.

P(1): 1 = 1(2 x 1 – 1) = 1, which is true.

Hence, P(l) is true.

Let us assume that P(n) is true for some natural number n = k.

∴ P(k):l+5 + 9 +…+(4k-3) = k(2k-1) (i)

Now, we have to prove that P(k + 1) is true.

P(k+ 1): 1 + 5 + 9 + … + (4k- 3) + [4(k+ 1) – 3]

$= 2k^2 -k+4k+ 4-3$

$= 2k^2 + 3k + 1$

= (k+ 1)(2k + 1)

= (k+l)[2(k+l)-l]

Hence, P(k + 1) is true whenever P(k) is true.

So, by the principle of mathematical induction P(n) is true for any natural number n.

Long Answer Type Questions

Q17. A sequence a_x, a_2, a_3, … is defined by letting $a_1=3$ and $a_k = 7a_{k-1}$ for all natural numbers k≥ Show that $a_n = 3 \cdot 7^{n-1}$ for all natural numbers.

Sol: We have a sequence a_x, a_2, a_3… defined by letting a, = 3 and $a_k = 7a_{k-1}$, for all natural numbers k≥2.

Let $P(n) : a_n = 3 \cdot 7^{n-1}$ for all natural numbers.

For $n = 2$, $a_2 = 3 \cdot 7^{2-1} = 3.7^1 = 21$

Also, $a_1 = 3$, $a_k = 7a_{k-1}$

$\Rightarrow \quad a_2 = 7 \cdot a_1 = 7 \times 3 = 21$

Thus, $P(2)$ is true.

Now, let us assume that $P(n)$ is true for some natural number $n = m$.

$\therefore \quad P(m) : a_m = 3 \cdot 7^{m-1} \quad$ (i)

Now, to prove that $P(m + 1)$ is true, we have to show that

$$P(m+1) : a_{m+1} = 3 \cdot 7^{m+1-1}$$

$$a_{m+1} = 7 \cdot a_{m+1-1} \text{ (as } a_k = 7a_{k-1})$$

$$= 7 \cdot a_m$$

$$= 7 \cdot 3 \cdot 7^{m-1} = 3 \cdot 7^{m-1+1} \quad \text{[Using (i)]}$$

Hence, $P(m + 1)$ is true whenever $P(m)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for any natural number n.

Q18. A sequence b_0, b_1, b_2, … is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$, for all natural numbers Show that $b_n = 5 + 4n$, for all natural number n using mathematical induction.

Sol. We have a sequence b_0, b_1, b_2,… defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$,, for all natural numbers k.

Sol. We have a sequence $b_0, b_1, b_2, \ldots$ defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$, for all natural numbers k.

Let $P(n) : b_n = 5 + 4n$, for all natural numbers

For $n = 1$, $b_1 = 5 + 4 \times 1 = 9$

Also $b_0 = 5$

$\therefore \quad b_1 = 4 + b_0 = 4 + 5 = 9$

Thus, $P(1)$ is true.

Now, let us assume that $P(n)$ is true for some natural number $n = m$.

$\therefore \quad P(m) : b_m = 5 + 4m \qquad$ (i)

Now, to prove that $P(k + 1)$ is true, we have to show that

$$P(m + 1) : b_{m+1} = 5 + 4(m + 1)$$

$$b_{m+1} = 4 + b_{m+1-1} \qquad (\text{As } b_k = 4 + b_{k-1})$$

$$= 4 + b_m$$

$$= 4 + 5 + 4m = 5 + 4(m + 1) \qquad [\text{Using (i)}]$$

Hence, $P(m + 1)$ is true whenever $P(m)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for any natural number n.

19. A sequence $d_1, d_2, d_3, \ldots$ is defined by letting $d_1 = 2$ and $d_k = \dfrac{d_{k-1}}{k}$, for all natural numbers $k \geq 2$. Show that $d_n = \dfrac{2}{n!}$, for all $n \in N$.

Sol. We have a sequence $d_1, d_2, d_3, \ldots$ defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$.

Let $P(n) : d_n = \frac{2}{n!}\ \forall\ n \in N$

$$P(2): d_2 = \frac{2}{2!} = \frac{2}{2 \times 1} = 1$$

Also, $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$

$$\Rightarrow \qquad d_2 = \frac{d_1}{2} = \frac{2}{2} = 1$$

Hence, $P(2)$ is true.

Now, let us assume that $P(n)$ is true for some natural number $n = m$.

$$\therefore \qquad P(m) : d_m = \frac{2}{m!} \qquad \text{(i)}$$

Now, to prove that $P(m + 1)$ is true, we have to show that

$$P(m+1): d_{m+1} = \frac{2}{(m+1)!}$$

$$d_{m+1} = \frac{d_{m+1-1}}{m+1} = \frac{d_m}{m+1} = \frac{2}{m!(m+1)} = \frac{2}{(m+1)!}$$

Hence, $P(m + 1)$ is true whenever $P(m)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for any natural number n.

20. Prove that for all $n \in N$,

$$\cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos[\alpha + (n-1)\beta]$$

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Sol. Let $P(n) : \cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \ldots + \cos [\alpha + (n-1)\beta]$

$$= \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Now, $P(1) : \cos \alpha = \dfrac{\cos\left[\alpha + \left(\frac{1-1}{2}\right)\beta\right]\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \dfrac{\cos \alpha \sin\frac{\beta}{2}}{\sin\frac{\beta}{2}} = \cos \alpha$

Hence, $P(1)$ is true.

Now, let us assume that $P(n)$ is true for some natural number $n = k$.

$\therefore \quad P(k) : \cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \ldots + \cos [\alpha + (k-1)\beta]$

$$= \frac{\cos\left[\alpha + \left(\frac{k-1}{2}\right)\beta\right]\sin\frac{k\beta}{2}}{\sin\frac{\beta}{2}} \qquad \text{(i)}$$

Now, to prove that $P(k+1)$ is true, we have to show that

$P(k+1) : \cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \ldots + \cos [\alpha + (k-1)\beta]$
$+ \cos[\alpha + (k+1-1)\beta]$

$$= \frac{\cos\left(\alpha + \frac{k\beta}{2}\right)\sin\frac{(k+1)\beta}{2}}{\sin\frac{\beta}{2}}$$

$\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \ldots + \cos [\alpha + (k-1)\beta] + \cos (\alpha + k\beta)$

$$= \frac{\cos\left[\alpha + \left(\frac{k-1}{2}\right)\right]\sin\frac{k\beta}{2}}{\sin\frac{\beta}{2}} + \cos(\alpha + k\beta) \qquad \text{[Using (i)]}$$

$$= \frac{\cos\left[\alpha + \left(\frac{k-1}{2}\right)\beta\right]\sin\frac{k\beta}{2} + \cos(\alpha + k\beta)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{k\beta}{2} - \frac{\beta}{2} + \frac{k\beta}{2}\right) - \sin\left(\alpha + \frac{k\beta}{2} - \frac{\beta}{2} - \frac{k\beta}{2}\right) + \sin\left(\alpha + k\beta + \frac{\beta}{2}\right) - \sin\left(\alpha + k\beta - \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + k\beta + \frac{\beta}{2}\right) - \sin\left(\alpha - \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{2\cos\frac{1}{2}\left(\alpha + k\beta + \frac{\beta}{2} + \alpha - \frac{\beta}{2}\right)\sin\frac{1}{2}\left(\alpha + k\beta + \frac{\beta}{2} - \alpha + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\cos\left(\frac{2\alpha + k\beta}{2}\right)\sin\left(\frac{k\beta + \beta}{2}\right)}{\sin\frac{\beta}{2}} = \frac{\cos\left(\alpha + \frac{k\beta}{2}\right)\sin(k+1)\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for any natural number n.

21. *Prove that* $\cos\theta\cos 2\theta\cos 2^2\theta\cos 2^{n-1}\theta = \dfrac{\sin 2^n\theta}{2^n\sin\theta}, \forall\, n \in N.$

Sol. Let $P(n) : \cos\theta\cos 2\theta .. \cos 2^{n-1}\theta = \dfrac{\sin 2^n\theta}{2^n\sin\theta}$

$$P(1) : \cos\theta = \frac{\sin 2^1\theta}{2^1\sin\theta} = \frac{\sin 2\theta}{2\sin\theta} = \frac{2\sin\theta\cos\theta}{2\sin\theta} = \cos\theta, \text{ which is true.}$$

Hence, $P(1)$ is true.

Now, let us assume that $P(n)$ is true for some natural number $n = k$.

$$\therefore P(k) : \cos\theta\cos 2\theta\cos 2^2\theta\ldots\cos 2^{k-1}\theta = \frac{\sin 2^k\theta}{2^k\sin\theta} \qquad \text{(i)}$$

To prove that $P(k + 1)$ is true, we have to show that

$$P(\mathrm{k}+1) : \cos\theta\cos 2\theta\cos 2^2\theta\cos 2^{k-1}\theta\cos 2^k\theta = \frac{\sin 2^{k+1}\theta}{2^{k+1}\sin\theta}$$

Now $\cos\theta\cos 2\theta\cos 2^2\theta\cos 2^{k-1}\theta\cos 2^k\theta$

$$= \frac{\sin 2^k\theta}{2^k\sin\theta}\cos 2^k\theta \qquad \text{[Using (i)]}$$

$$= \frac{2\sin 2^k\theta\cos 2^k\theta}{2\cdot 2^k\sin\theta}$$

$$= \frac{\sin 2\cdot 2^k\theta}{2^{k+1}\sin\theta} = \frac{\sin 2^{(k+1)}\theta}{2^{k+1}\sin\theta}$$

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

22. Prove that, $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \dfrac{\sin\dfrac{n\theta}{2}\sin\dfrac{(n+1)}{2}\theta}{\sin\dfrac{\theta}{2}}$, for all $n \in N$.

Sol. Let $P(n)$: $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \dfrac{\sin\dfrac{n\theta}{2}\sin\dfrac{(n+1)}{2}\theta}{\sin\dfrac{\theta}{2}}$, for all $n \in N$

$$P(1):\sin\theta = \frac{\sin\frac{\theta}{2}\cdot\sin\frac{(1+1)}{2}\theta}{\sin\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}\cdot\sin\theta}{\sin\frac{\theta}{2}} = \sin\theta$$

Hence, $P(1)$ is true.

Now, let us assume that $P(n)$ is true for some natural number $n = k$.

$$\therefore\ P(k) : \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin k\theta = \frac{\sin\frac{k\theta}{2}\sin\left(\frac{k+1}{2}\right)\theta}{\sin\frac{\theta}{2}} \qquad \text{(i)}$$

Now, to prove that $P(k+1)$ is true, we have to show that

$P(k+1)$: $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin k\theta + \sin(k+1)\theta$

$$= \frac{\sin\frac{(k+1)\theta}{2}\sin\left(\frac{k+1+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

$\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin k\theta + \sin(k+1)\theta$

$$= \frac{\sin\frac{k\theta}{2}\sin\left(\frac{k+1}{2}\right)\theta}{\sin\frac{\theta}{2}} + \sin(k+1)\theta \qquad \text{[Using(i)]}$$

$$= \frac{\sin\frac{k\theta}{2}\sin\left(\frac{k+1}{2}\right)\theta + \sin(k+1)\theta\cdot\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \frac{\cos\left[\frac{k\theta}{2} - \left(\frac{k+1}{2}\right)\theta\right] - \cos\left[\frac{k\theta}{2} + \left(\frac{k+1}{2}\right)\theta\right] + \cos\left[(k+1)\theta - \frac{\theta}{2}\right] - \cos\left[(k+1)\theta + \frac{\theta}{2}\right]}{2\sin\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2} - \cos\left(k\theta + \frac{\theta}{2}\right) + \cos\left(k\theta + \frac{\theta}{2}\right) - \cos\left(k\theta + \frac{3\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2} - \cos\left(k\theta + \frac{3\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{2\sin\frac{1}{2}\left(\frac{\theta}{2} + k\theta + \frac{3\theta}{2}\right) \cdot \sin\frac{1}{2}\left(k\theta + \frac{3\theta}{2} - \frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left(\frac{k\theta + 2\theta}{2}\right) \cdot \sin\left(\frac{k\theta + \theta}{2}\right)}{\sin\frac{\theta}{2}}$$

$$= \frac{\sin(k+1)\frac{\theta}{2} \cdot \sin(k+1+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for any natural number n.

23. Show that $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is a natural number, for all $n \in N$.

Sol. Let $P(n)$: $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$ is a natural number, for all $n \in N$.

$P(1) : \frac{1^5}{5}+\frac{1^3}{3}+\frac{7(1)}{15}=\frac{3+5+7}{15}=\frac{15}{15}=1,$ which is a natural number.

Hence, $P(1)$ is true.

Let us assume that $P(n)$ is true, for some natural number $n = k$.

$\therefore \quad P(k) : \frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}$ is natural number (i)

Now, we have to prove that $P(k + 1)$ is true.

$$P(k+1): \frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{7(k+1)}{15}$$

$$=\frac{k^5+5k^4+10k^3+10k^2+5k+1}{5}+\frac{k^3+1+3k^2+3k}{3}+\frac{7k+7}{15}$$

$$=\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}+\frac{5k^4+10k^3+10k^2+5k+1}{5}+\frac{3k^2+3k+1}{3}+\frac{7}{15}$$

$$=\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}+k^4+2k^3+2k^2+k+k^2+k+\frac{1}{5}+\frac{1}{3}+\frac{7}{15}$$

$$=\frac{k^5}{5}+\frac{k^3}{3}+\frac{7k}{15}+k^4+2k^3+3k^2+2k+1$$

which is a natural number **[Using (i)]**

Hence, $P(k + 1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for any natural number n.

24. Prove that $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}>\frac{13}{24}$, for all natural numbers $n>1$.

Sol. Let $P(n)$: $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}>\frac{13}{24}$, for all natural numbers $n>1$.

$$\Rightarrow \quad P(2): \frac{1}{2+1}+\frac{1}{2+2}>\frac{13}{24}$$

$$\frac{1}{3}+\frac{1}{4}>\frac{13}{24} \Rightarrow \frac{4+3}{12}>\frac{13}{24}$$

$$\Rightarrow \quad \frac{7}{12}>\frac{13}{24}, \text{ which is true.}$$

Hence, $P(2)$ is true.

Let us assume that $P(n)$ is true, for some natural number $n=k$.

$$\therefore \quad P(k): \frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}>\frac{13}{24} \qquad \text{(i)}$$

Now, to prove that $P(k+1)$ is true, we have to show that

$$P(k+1): \frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}+\frac{1}{2(k+1)}>\frac{13}{24}$$

$$\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{2k}+\frac{1}{2(k+1)}>\frac{13}{24}+\frac{1}{2(k+1)}>\frac{13}{24}$$

$$\left(\because \frac{1}{2(k+1)}>0\right)$$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction P(n) is true for any natural number rt,n> 1.

Q25. Prove that number of subsets of a set containing n distinct elements is 2", for all n ∈

Sol: Let P(n): Number of subset of a set containing n distinct elements is 2", for all ne N.

For n = 1, consider set A = {1}. So, set of subsets is {{1}, ∅}, which contains 2^1 elements.

So, P(1) is true.

Let us assume that P(n) is true, for some natural number n = k.

P(k): Number of subsets of a set containing k distinct elements is 2^k To prove that P(k + 1) is true, we have to show that P(k + 1): Number of subsets of a set containing (k + 1) distinct elements is 2^{k+1}

We know that, with the addition of one element in the set, the number of subsets become double.

Number of subsets of a set containing (k+ 1) distinct elements = $2 \times 2^k = 2^{k+1}$

So, P(k + 1) is true. Hence, P(n) is true.